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COMFORMALITY AND ISOMETRY OF RIEMANNIAN MANIFOLDS TO SPHERES. II

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1. Introduction

Let M be an n-dimensional ($n \geq 2$) connected smooth Riemannian manifold with positive definite metric g. If a vector field v on M defines an infinitesimal conformal transformation on (M, g), then v satisfies $\mathscr{L}_v g = 2\rho g$ where $\mathscr{L}_v$ denotes the Lie derivative with respect to v, and ρ is a function on M. v defines an infinitesimal homothetic transformation or infinitesimal isometry according as ρ is constant or zero.

In the last decade or so several authors (for exhaustive lists see [7], [9]) have studied conditions for a Riemannian manifold of dimension $n \geq 2$ with constant scalar curvature k to be either conformal or isometric to a sphere. Recently Ackerman and Hsiung [1], Yano and Hiramatu [7], [8] and Amur and Pujar [2] have studied the conditions without putting restrictions on the scalar curvature k such as $\mathscr{L}_v k = 0$, $\mathscr{L}_{D\rho}\mathscr{L}_v k = 0$ or $[v, D\rho]k = 0$, etc. where $D\rho$ is the vector field on M associated with the differential 1-form $d\rho$.

In this paper we consider a metric semi-symmetric connection $\mathring{\nabla}$ on M induced by a smooth function ρ on M, and obtain conditions for M to be conformal or isometric to a sphere. It is shown in § 5 that our results include some results of Yano and Obata [9] and some of Hsiung and Mugridge [3] as special cases.

2. Notation and formulas

Let ∇ denote a Riemannian connection on M. If x^i, $i = 1, 2, \cdots, n$, are local coordinates in a neighborhood of a point x of M, then the Christoffel symbols associated with ∇ are denoted by $\left\{ {i \atop j\ k} \right\}$, and the components of g by g_{ij}. The raising and lowering of the indices are as usual carried out respectively with g^{ij} and g_{ij}. Let ρ be a smooth function of M. Then $\pi = d\rho$ is *a* smooth closed differential 1-form on M. The local components of π will be denoted by ρ_i. A

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connection $\mathring{\nabla}$ on M, whose Christoffel symbols are denoted by Γ^h_{jk}, is defined by

$$\Gamma^h_{jk} = \left\{ {h \atop j\ k} \right\} + \delta^h_j \rho_k - g_{kj} \rho^h \tag{2.1}$$

where $\rho^h = g^{hi}\rho_i$. Since $\mathring{\nabla}_i g_{kj} = 0$ holds and Γ^h_{jk} is not symmetric, the connection $\mathring{\nabla}$ is called a metric semi-symmetric connection on M, [6].

The components $\mathring{K}_{kji}{}^h$ of the curvature tensor $\mathring{K}$ of $\mathring{\nabla}$ and $K_{kji}{}^h$ of the curvature tensor K of ∇ are related by

$$\mathring{K}_{kji}{}^h = K_{kji}{}^h - \alpha_{ji}\delta^h_k + \alpha_{ki}\delta^h_j - g_{ji}A^h_k + g_{ki}A^h_j \,, \tag{2.2}$$

where

$$\alpha_{ji} = \nabla_j \rho_i - \rho_j \rho_i + \tfrac{1}{2} g_{ij} \rho_k \rho^k \tag{2.3}$$

are components of a tensor field of type (0, 2) on M and $A^h_j = g^{hi}\alpha_{ij}$. (2.2) shows that we can regard $\mathring{K}$ as a tensor field on the Riemannian space (M, g). Setting $\mathring{K}_{kjih} = g_{hl}\mathring{K}_{kji}{}^l$ we have

$$\mathring{K}_{jkih} = -\mathring{K}_{kjih} \,, \qquad \mathring{K}_{kjhi} = -\mathring{K}_{kjih} \,. \tag{2.4}$$

Since π is a closed 1-form on M, it follows that α_{ji} is symmetric in i and j, consequently $\mathring{K}$ satisfies Bianchi first identity. Hence we obtain [4]

$$\mathring{K}_{ihkj} = \mathring{K}_{kjih} \,. \tag{2.5}$$

Contracting (2.2) with respect to the indices h and k we have

$$\mathring{K}_{ji} = K_{ji} - (n-2)\alpha_{ji} - \alpha g_{ij} \,, \tag{2.6}$$

where

$$\alpha = g^{ji}\alpha_{ji} = \nabla_i \rho^i + \frac{n-2}{2}\rho_k \rho^k \,. \tag{2.7}$$

Transvection of (2.6) with g^{ji} yields

$$\mathring{k} = k - 2(n-1)\alpha \,, \tag{2.8}$$

where $\mathring{k} = g^{ij}\mathring{K}_{ij}$.

We define a positive smooth function u on M by setting

$$u(x) = e^{-\rho(x)} \tag{2.9}$$

for all $x \in M$. Denoting the covariant differentiation of u with respect to ∇_i by u_i, we have

$$(2.10)\quad \begin{aligned} &\text{(i)}\quad u_i = -u\rho_i\,, \qquad \text{(ii)}\quad \nabla_j u_i = u(\rho_j\rho_i - \nabla_j\rho_i)\,, \\ &\text{(iii)}\quad \Delta u = u(\rho^k\rho_k - \Delta\rho)\,, \end{aligned}$$

where $\Delta = g^{ij}\nabla_j\nabla_i$ is the Laplacian operator.

Now from (2.7), (2.8) and (2.10) (iii) we obtain

$$(2.11)\quad u^2(\mathring{k} - k) = 2(n-1)u\Delta u - n(n-1)u_i u^i\,.$$

Corresponding to the tensor fiields G, Z and W (for definitions see [7], [3]) on (M, g) we define $\mathring{G}$, $\mathring{Z}$ and $\mathring{W}$ on the same space by

$$(2.12)\quad \mathring{G}_{ij} = \mathring{K}_{ij} - \frac{\mathring{k}}{n}g_{ij}\,,$$

$$(2.13)\quad \mathring{Z}_{kjih} = \mathring{K}_{kjih} - \frac{\mathring{k}}{n(n-1)}(g_{hk}g_{ij} - g_{hj}g_{ki})\,,$$

$$(2.14)\quad \begin{aligned} \mathring{W}_{kjih} &= a\mathring{Z}_{kjih} + b_1 g_{kh}\mathring{G}_{ji} - b_2 g_{ki}\mathring{G}_{jh} + b_3 g_{ji}\mathring{G}_{kh} \\ &\quad - b_4 g_{jh}\mathring{G}_{ki} + b_5 g_{kj}\mathring{G}_{ih} - b_6 g_{ih}\mathring{G}_{kj}\,, \end{aligned}$$

where $a, b_1, \cdots, b_6$ are the same constants which occur in the definition of W_{kjih}.

Substituting for $\mathring{K}_{ij}$ and $\mathring{k}$ from (2.6) and (2.8) respectively in (2.12) we obtain

$$(2.15)\quad \mathring{G}_{ji} = G_{ji} + (n-2)T_{ji}\,,$$

where

$$(2.16)\quad \begin{aligned} T_{ji} &= (\rho_j\rho_i - \nabla_j\rho_i) + \frac{1}{n}(\rho^k\rho_k - \Delta\rho)g_{ji} \\ &= u^{-1}\left(\nabla_j u_i - \frac{1}{n}\Delta u g_{ji}\right) \end{aligned}$$

in view of (2.10). It is easy to see that

$$(2.17)\quad g^{ij}T_{ij} = 0\,, \qquad g^{ij}G_{ij} = 0\,.$$

Computations similar to those for $\mathring{G}_{ji}$ yield

$$(2.18)\quad \mathring{Z}_{kjih} = Z_{kjih} + S_{kjih}\,,$$

where

$$S_{kjih} = g_{kh}T_{ji} - g_{jh}T_{ki} + T_{kh}g_{ji} - T_{jh}g_{ki} \,, \tag{2.19}$$

$$\mathring{W}_{kjih} = W_{kjih} + Q_{kjih} \,, \tag{2.20}$$

where

$$\begin{aligned} \frac{Q_{kjih}}{n-2} &= \left(\frac{a}{n-2} + b_1\right)g_{kh}T_{ji} - \left(\frac{a}{n-2} + b_2\right)g_{ki}T_{jh} \\ &\quad + \left(\frac{a}{n-2} + b_3\right)g_{ji}T_{kh} - \left(\frac{a}{n-2} + b_4\right)g_{jh}T_{ki} \\ &\quad + b_5 g_{kj}T_{ih} - b_6 g_{ih}T_{kj} \,. \end{aligned} \tag{2.21}$$

It is easy to see that

$$T_{ij}T^{ij} = u^{-2}\left(\nabla^j u^i - \frac{1}{n}\Delta u g^{ij}\right)\left(\nabla_j u_i - \frac{1}{n}\Delta u g_{ij}\right) , \tag{2.22}$$

$$\mathring{G}_{ij}\mathring{G}^{ij} = G_{ij}G^{ij} + 2(n-2)G_{ij}T^{ij} + (n-2)^2 T_{ij}T^{ij} \,, \tag{2.23}$$

$$\mathring{W}_{kjih}\mathring{W}^{kjih} = W_{kjih}W^{kjih} + 2c(n-2)T_{ij}G^{ij} + c(n-2)^2 T_{ij}T^{ij} \,, \tag{2.24}$$

where c is a constant given by [3]

$$\begin{aligned} c &= \frac{4a^2}{n-2} + 2a\sum_{i=1}^{6} b_i + \left(\sum_{i=1}^{6} (-1)^{i-1} b_i\right)^2 + (n-1)\sum_{i=1}^{6} b_i^2 \\ &\quad - 2(b_1b_2 + b_2b_4 - b_5b_6) \,. \end{aligned} \tag{2.25}$$

3. Lemmas

Lemma 3.1. *Suppose M is orientiable and compact. $\rho =$ constant if and only if the scalar function $\mathring{k}$ is equal to the scalar curvarure k of M.*

Proof. If $\rho =$ constant, it is trivial to see that $\mathring{k} = k$. Suppose $\mathring{k} = k$ holds. Then from (2.11) we have $2u\Delta u - nu_iu^i = 0$ which implies

$$\int_M u^{-1}(u_iu^i)dV = 0 \,,$$

where dV denotes volume element of M. Since $u > 0$, the integral equation implies $u =$ constant which in view of (2.9) implies $\rho =$ constant.

Lemma 3.2. *Suppose M is compact and orientiable. Then*

$$\int_M (\nabla^j u^i)G_{ji}dV = -\frac{n-2}{2n}\int_M u^i\nabla_i k dV = -\frac{n-2}{2n}\int_M \mathscr{L}_{Du}k dV \,, \tag{3.1}$$

where Du is the vector field on M associated with the 1-form du.

Proof. Since $\nabla^j K_{ji} = \frac{1}{2}\nabla_i k$, from the formula $G_{ij} = K_{ij} - (k/n)g_{ij}$ it follows that

$$\nabla^j G_{ji} = \frac{n-2}{2n}\nabla_i k \ . \tag{3.2}$$

Hence by directly computing $\nabla^j(u^i G_{ij})$, using (3.2) and integrating over M we obtain (3.1).

Lemma 3.3. *Suppose M is orientable and compact. Then the following integral formulas hold for M:*

$$\frac{1}{n}\int_M [nu(\mathring{G}_{ij}\mathring{G}^{ij} - G_{ij}G^{ij}) + (n-2)^2 \mathscr{L}_{Du}k - n(n-2)^2 u T_{ji}T^{ji}]dV = 0 \ , \tag{3.3}$$

$$\begin{aligned}\frac{1}{n}\int_M [nu(\mathring{W}_{kjih}\mathring{W}^{kjih} - W_{kjih}W^{kjih}) + c(n-2)^2 \mathscr{L}_{Du}k \\ - n(n-2)^2 cu T_{ij}T^{ij}]dV = 0 \ . \end{aligned}\tag{3.4}$$

Proof. Since $g^{ij}G_{ij} = 0$, from (2.17) we can write (2.23) in the form

$$u\{\mathring{G}_{ij}\mathring{G}^{ij} - G_{ij}G^{ij} - (n-2)^2 T_{ij}T^{ij}\} = 2(n-2)G_{ij}\nabla^j u^i \ .$$

On integrating over M and using Lemma 3.2, we obtain (3.3). The proof of (3.4) is similar.

To prove the next lemma we need the following known theorem.

Theorem A (*Tashiro* [5]). *If a compact Riemannian manifold M of dimension $n \geq 2$ admits a nonconstant function ρ such that*

$$\nabla^j \nabla^i \rho = \frac{1}{n}\Delta \rho g_{ij} \ , \tag{3.5}$$

then M is conformal to a sphere.

Lemma 3.4. *Suppose M of dimension $n \geq 2$ is compact, and admits a nonconstant function ρ. M is conformal to a sphere if the tensor field with components T_{ij} is identically zero on M.*

Proof. Since $u > 0$, from the expression (2.10) for T_{ij} it follows that $T_{ij} = 0$ if and only $\nabla_i u_j = \Delta u g_{ij}/n$. Hence from Theorem A the required result follows.

Finally we list a lemma due to Yano and Obata [9].

Lemma 3.5. *Suppose M of dimension $n \geq 2$ is complete. If $\mathscr{L}_{Du}k = 0$ and $\nabla_i \nabla_j u = \Delta u g_{ij}/n$ holds for a nonconstant function u, then M is isometric to a sphere.*

4. Thoerems

Throughout this and the next sections we shall assume that M is a compact orientable smooth Riemannian manifold of dimension $n > 2$.

Theorem 4.1. *Let ρ be a smooth function on M and $u = e^{-\rho}$. Then*

$$\int_M [nu(\mathring{G}_{ij}\mathring{G}^{ij} - G_{ij}G^{ij}) + (n-2)\mathscr{L}_{Du}k]dV \geq 0 , \tag{4.1}$$

$$\int_M [nu(\mathring{W}_{kjih}\mathring{W}^{kjih} - W_{kjih}W^{kijh}) + c(n-2)^2\mathscr{L}_{Du}k]dV \geq 0 , \quad (c > 0) , \tag{4.2}$$

where the tensors $\mathring{G}$ and $\mathring{W}$ are formed with the help of the metric semi-symmetric connection induced by ρ. If ρ is such that the equality in integral equation (4.1) or (4.2) holds, then M is conformal to a sphere.

Proof. Follows from Lemmas 3.3 and 3.4.

Theorem 4.2. *If a smooth nonconstant function ρ on M is such that*

$$\mathscr{L}_{Du}k = 0 , \qquad \mathring{G}_{ij}\mathring{G}^{ij} = G_{ij}G^{ij} , \tag{4.3}$$

or such that

$$\mathscr{L}_{Du}k = 0 , \quad \mathring{W}_{kjih}\mathring{W}^{kjih} = W_{kjih}W^{kjih} , \quad (c > 0) , \tag{4.4}$$

then M is isometric to a sphere.

Proof. Follows from Lemmas 3.3 and 3.5 and the conditions stated in the theorem.

Theorem 4.3. *Suppose M is an Einstein manifold. If a smooth nonconstant function ρ on M is such that*

$$\mathring{G}_{ij} = 0 \tag{4.5}$$

or such that

$$\mathring{W}_{kjih} = 0 , \qquad (c > 0) , \tag{4.6}$$

then M is isometric to a sphere.

Proof. For an Einstein manifold $G_{ij} = 0$. Hence from Lemmas 3.3 and 3.5 and the conditions stated in the theorem the result follows.

5. Special cases

(i) Let ρ be a smooth function on M arising from a conformal change of metric on M, that is, let ρ be such that a metric g^* on M is conformally related to g by

$$g^*_{ij} = e^{2\rho} g_{ij} \,. \tag{5.1}$$

For any tensor with respect to g, the corresponding tensor with respect to g^* will be denoted by the same letter with a star. The function ρ induces a metric semi-symmetric connection $\mathring{\nabla}$ on M and a connection ∇^*, called the conformal change of connection on M. The expressions for the curvature tensors $\mathring{K}$ and K^* in terms of K and the derivatives of ρ with respect to the Riemannian connection ∇ are the same (see [4]). Since $g^{*ij} = e^{-2\rho} g_{ij}$, we have

$$K^*_{kji}{}^h = \mathring{K}_{kji}{}^h \,, \quad K^*_{ji} = \mathring{K}_{ji} \,, \quad k^* = e^{-2\rho}\mathring{k} \,, \tag{5.2}$$

so that

$$G^*_{ji} = \mathring{G}_{ji} \,, \quad Z^*_{kji}{}^h = \mathring{Z}_{kji}{}^h \,, \quad W^*_{kji}{}^h = \mathring{W}_{kji}{}^h \,. \tag{5.3}$$

It is easy to see that

$$G^{*ij} = e^{-4\rho}\mathring{G}^{ij} \,, \tag{5.4}$$

$$W^*_{kjih} = e^{2\rho}\mathring{W}_{kjih} \,, \qquad W^{*kjih} = e^{-6\rho}\mathring{W}^{kjih} \,, \tag{5.5}$$

so that

$$G^{*ij}G^*_{ij} = e^{-4\rho}\mathring{G}_{ij}\mathring{G}^{ij} = u^{4\rho}\mathring{G}_{ij}\mathring{G}^{ij} \,, \tag{5.6}$$

$$W^*_{kjih}W^{*kjih} = e^{-4\rho}\mathring{W}_{kjih}\mathring{W}^{kjih} = u^{4\rho}\mathring{W}_{kjih}\mathring{W}^{kjih} \,, \tag{5.7}$$

where $u = e^{-\rho}$.

Substituting (5.6) in (3.3) we obtain

$$\int_M \left[(u^{-3}G^*_{ij}G^{*ij} - uG_{ij}G^{ij}) + \frac{1}{n}(n-2)^2\mathscr{L}_{Du}k - (n-2)^2 uT_{ij}T^{ij} \right] dV = 0 \,, \tag{5.8}$$

which is an integral formula due to Yano and Obata [9].

Again substituting (5.7) in (3.4) we have

$$\int_M \left[(u^{-3}W^*_{kjih}W^{*kjih} - uW_{kjih}W^{kjih}) + \frac{1}{n}c(n-2)^2\mathscr{L}_{Du}k - (n-2)^2 cuT_{ij}T^{ji} \right] dV = 0 \,, \tag{5.9}$$

which is a formula due to Hsiung and Mugridge [3].

(ii) Suppose ρ is a smooth function on M satisfying

$$K_{kjih} = e^{-\rho}\{\alpha_{ji}g_{hk} - \alpha_{ki}g_{hj} + g_{ji}\alpha_{hk} - g_{ki}\alpha_{hj}\} \,. \tag{5.10}$$

Then it follows from (2.2) that

(5.11) $\mathring{K}_{kjih} = (1 - u^{-1})K_{kjih}\,, \quad \mathring{K}_{ji} = (1 - u^{-1})K_{ji}\,, \quad \mathring{k} = (1 - u^{-1})k\,,$

so that

(5.12) $\mathring{G}_{ji} = (1 - u^{-1})G_{ji}\,, \qquad \mathring{W}_{kjih} = (1 - u^{-1})W_{kjih}\,.$

For this special case, (4.1) and (4.2) reduce to

(5.13) $$\int_M \left[n(u^{-1} - 2)G_{ij}G^{ij} + (n - 2)\mathscr{L}_{Du}k\right]dV \geq 0\,,$$

(5.14) $$\int_M \left[n(u^{-1} - 2)W_{kjih}W^{kjih} + c(n - 2)^2\mathscr{L}_{Du}k\right]dV \geq 0\,.$$

Thus, if ρ is a nonconstant smooth function on M satisfying (5.10) and is such that the equality in (5.13) or (5.14) holds, then M is conformal to a sphere.

On the other hand, if M is Einsteinian and ρ is a nonconstant function satisfying (5.10), then, since $\mathring{G}_{ij} = (1 - u^{-1})G_{ij}$, it follows that $\mathring{G}_{ij} = 0$. Theorem 4.3 shows that M is isometric to a sphere.

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